Virial theorem: applications

(1) Estimation of internal temperature

Set $\ddot{I} = 0$, $T = 0$. Then,

$$3(\gamma - 1)U = -\Omega$$

Estimating the terms for an ideal monatomic gas, with $\gamma = 5/3$:

$$U \simeq \frac{3}{2}kT\frac{MN_A}{\mu}$$

$$\Omega \sim -\frac{GM^2}{R}$$

which gives for $\mu = 0.6$,

$$T \sim 5 \times 10^6 \left(\frac{M}{M_\odot}\right)\left(\frac{R}{R_\odot}\right)^{-1} \text{ K}$$

Implications:

- $T \gg T_e$ – i.e. strong temperature gradient
- Most of the star is highly ionized
- Central temperature is to order of magnitude sufficient for nucleosynthesis

(2) Total net energy

The total net energy of the star is,

$$W = U + \Omega$$

For hydrostatic stars with a γ law equation of state the virial theorem gives,

$$W = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega = -(3\gamma - 4)U$$

Note:

(i) Since $\Omega < 0$, $W < 0$ iff $\gamma > 4/3$. So a star with $\gamma = 5/3$ is safely bound. But for radiation pressure, $P = \frac{1}{3}aT^4$, and energy per unit volume is aT^4. ie:

$$P = (\gamma - 1)u$$

with $\gamma = 4/3$. Thus as radiation pressure becomes more dominant, stars become less gravitationally bound.

(ii) Gravitational contraction. As star radiates into space $\dot{W} < 0$,

$$\rightarrow \dot{\Omega} < 0, \quad \dot{U} > 0$$

i.e. in the absence of energy sources other than gravity, as a star radiates it contracts and heats up (negative specific heat).

(3) Timescale for pulsations

Consider a star executing small amplitude radial pulsations. Assume that during the pulsation, the structure is homologous (i.e. that $m(r/R)$ is a fixed function). Then,

$$
\begin{aligned}
I &= sMR^2 \\
\Omega &= -q\frac{GM^2}{R}
\end{aligned}
$$

with s, q dimensionless constants. Let,

$$R(t) = R_0 + \epsilon \Delta R(t)$$

with R_0 the equilibrium radius. Writing the virial theorem in the form,

$$\frac{1}{2}\ddot{I} = 3(\gamma - 1)W + (4 - 3\gamma)\Omega$$

then in equilibrium,

$$0 = 3(\gamma - 1)W + (4 - 3\gamma)\Omega_0$$

where W needs no subscript since we can assume that the total energy remains fixed during the pulsations.

Strategy: write virial equation to first order in ϵ to derive equation for rate of change of ΔR.

LHS:

$$\ddot{I} = 2sM(R\ddot{R} + \dot{R}^2)$$

$\dot{R}^2$ is second order in ϵ, so drop this term. Then,

$$\ddot{R} = \epsilon \frac{d^2 \Delta R}{dt^2}$$

giving,

$$\ddot{I} = 2sMR_0\epsilon \frac{d^2 \Delta R}{dt^2}$$

RHS:

$$\begin{aligned} \Omega = -q\frac{GM^2}{R} &= -q\frac{GM^2}{R_0 + \epsilon\Delta R(t)} \\ &= -q\frac{GM^2}{R_0}\left[1 - \epsilon\frac{\Delta R}{R_0}\right] \\ &= \Omega_0\left[1 - \epsilon\frac{\Delta R}{R_0}\right] \end{aligned}$$

Substituting in the virial theorem:

$$sMR_0\epsilon\frac{d^2\Delta R}{dt^2} = (3\gamma - 4)\Omega_0 + (4 - 3\gamma)\Omega_0 \left[1 - \epsilon\frac{\Delta R}{R_0}\right]$$

which simplifies to give,

$$\frac{d^2\Delta R}{dt^2} = (3\gamma - 4)\frac{\Omega_0}{I_0}\Delta R$$

Note:

- $\Omega_0 < 0$, so for stable oscillations we require $\gamma > 4/3$ (i.e. a bound star).
- In the oscillatory case,

$$P = 2\pi\left(\frac{sR^3}{(3\gamma - 4)qGM}\right)^{1/2}$$

- For $q = 3/2$, $s = 0.2$, $\gamma = 5/3$, and Solar mass and radius, find $P \simeq$ an hour.

Timescales

Dynamical timescale

Imagine that the pressure support in the star were suddenly removed. Outer layers of the star would then collapse with a velocity comparable to the escape velocity,

$$v^2 \sim \frac{GM}{R}$$

The dynamical timescale is thus,

$$t_{\rm dyn} \sim \frac{R}{v} \sim \sqrt{\frac{R^3}{GM}} \sim \frac{1}{[G < \rho >]^{1/2}}$$

where $< \rho >$ is a characteristic density. For the Sun, this is about 1600 s.

Note: If the structure is changing, compare the timescale of that change to $t_{\rm dyn}$ to determine if hydrostatic equilibrium remains a valid approximation.

Nuclear timescale

Time required to leave the main sequence due to burning of a 'significant' amount of nuclear fuel,

$$t_{\rm nuc} = \frac{\epsilon q_{SC} M c^2}{L}$$

where the efficiency of nuclear burning $\epsilon \simeq 0.007$ for the most important reaction H $\rightarrow$ He.

The factor $q_{SC} \approx 0.1$ accounts for the fact that the stellar core contracts and the star leaves the main sequence well before all the hydrogen is exhausted (Schönberg & Chandrasekhar).

For stars of Solar mass and above,

$$\frac{L}{L_\odot} \approx \left(\frac{M}{M_\odot}\right)^{3.5}$$

so the nuclear timescale is,

$$t_{\rm nuc} \sim 10^{10} \left(\frac{M}{M_\odot}\right)^{-2.5} \ {\rm yr}$$

i.e. more massive stars have shorter lives.

Numerical calculations give a main sequence lifetime of 480 Myr for 2.25 $M_\odot$, 20 Myr for 9 $M_\odot$.

Kelvin-Helmholtz timescale

Timescale for the star to radiate its present thermal (or gravitational) energy,

$$t_{\rm KH} \approx \frac{GM^2}{RL} \sim 3 \times 10^7 \ {\rm yr}$$

for Solar parameters.

How does this compare with the time required for a photon to random walk out of the star?

Anticipating somewhat results to be derived later, can show that $t_{\rm KH}$ is also the timescale on which the star will adjust globally to changes in the thermal structure. Combine the energy transport equation,

$$L_r = -\frac{16\pi acr^2}{3\kappa\rho}T^3\frac{\partial T}{\partial r} = -\frac{64\pi^2 acr^4}{3\kappa}T^3\frac{\partial T}{\partial m}$$

with a generalized version of the energy generation equation that allows for changes in the internal energy u and density ρ with time,

$$\frac{\partial L_r}{\partial m} = \epsilon - \frac{\partial u}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}$$

Combining these, and writing $u = c_V T$,

$$\frac{\partial}{\partial m}\left(\sigma^*\frac{\partial T}{\partial m}\right) = c_V\frac{\partial T}{\partial t} - \left[\epsilon + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\right]$$

If the second term on the RHS is ignored, then this is just a diffusion equation with diffusion coefficient σ_*/c_V, where,

$$\sigma^* = \frac{64\pi^2 acT^3r^4}{3\kappa}$$

Timescale for diffusion of thermal energy across a shell of thickness Δm is then,

$$t_{\rm adj} = \frac{(\Delta m)^2}{\sigma^*/c_V}$$

For the whole star, $\Delta m = M$, and,

$$L \sim \frac{\sigma^* T}{M}$$

where T and σ^* are now understood (for definiteness) to be average values. Then,

$$t_{\rm adj} \sim \frac{c_V T M}{L} \sim \frac{U}{L} \sim t_{KH}$$

i.e. the Kelvin-Helmholtz timescale is the time required for a thermal pulse to travel from center to surface.

Note:

- Because $t_{\rm adj} \ll t_{\rm nuc}$ for stars like the Sun on the main sequence, valid to assume that such stars are in both mechanical and thermal equilibrium.
- $t_{\rm adj}$ is useful quantity when considering thermal properties of small regions within the star – e.g. the core