Physical structure equations

The most general (Newtonian) forms of the continuity and momentum equations are just the usual hydrodynamic equations,

$$\begin{aligned}\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= -\nabla P - \rho\nabla\Phi \\ \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) &= 0 \\ \nabla^2\Phi &= 4\pi G\rho\end{aligned}$$

where,

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

- $\mathbf{v}$ is the fluid velocity
- ρ is density
- P is pressure
- Φ is the gravitational potential

In spherical symmetry these eqns reduce to:

$$\begin{aligned} \frac{\mathrm{d}m}{\mathrm{d}r} &= 4\pi r^2 \rho \\ \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} &= -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{Gm}{r^2} \end{aligned}$$

When can we ignore the terms on the LHS? Suppose the star's radius changes on some timescale τ. Then a characteristic velocity V is,

$$V \sim \frac{R}{\tau}$$

and the terms on the LHS are of magnitude,

$$v_r \frac{\partial v_r}{\partial r} \sim \frac{R}{\tau^2}$$

Comparing to the gravitational force,

$$\frac{GM}{R^2} = R\frac{GM}{R^3} = \frac{R}{t_{\mathrm{dyn}}^2}$$

Therefore ratio is $t_{\mathrm{dyn}}^2/\tau^2$ which is $\ll 1$ in most cases.

Equation of hydrostatic equilibrium

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{Gm}{r^2}$$

or using mass as the co-ordinate,

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}$$

Note: If $P = P(\rho)$ then hydrostatic equilibrium + continuity are sufficient to determine the structure. These models are 'polytropes', and can be useful even if P is **not** just a function of the density.

For many normal stars gas pressure and radiation are the important contributions,

$$\begin{aligned} P &= P_{\mathrm{gas}} + P_{\mathrm{rad}} \\ &= \frac{\rho R T}{\mu} + \frac{1}{3}aT^4 \end{aligned}$$

To determine μ need to know,

- The constituents of the plasma
- The state of ionization

Define:

$$\mu = \text{ average weight of particles } /m_H$$

e.g. For pure H, $\mu = 1$ (unionized), $\mu = 1/2$ (fully ionized).

Assume for the moment that the gas is fully ionized. Denote abundances of different elements as fraction per unit mass:

- X – hydrogen
- Y – helium
- Z – the rest, 'metals'

With this definition, $X + Y + Z = 1$.

Gas pressure:

$$P_{\text{gas}} = NkT$$

where,

$$N = n_x + n_y + n_Z + n_e$$

and e.g. n_e is the partial number density of electrons.

For most metals, the ratio of the nuclear charge Z_i to the atomic number A_i is roughly,

$$Z_i/A_i \approx 1/2$$

Hence,

	H	He	'metal'
No. density of nuclei	$X\rho/m_H$	$Y\rho/4m_H$	$Z\rho/Am_H$
No. density of electrons	$X\rho/m_H$	$2Y\rho/4m_H$	$\simeq (A/2)Z\rho/Am_H$

For $A \gg 1$,

$$N = \frac{\rho}{m_H}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]$$

$$\rightarrow \mu^{-1} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

Limits:

- $X = 1$, $\mu = 1/2$
- $Z = 1$, $\mu = 2$

Often, $Z \approx 0$, $X + Y \simeq 1$. In this case,

$$\mu = \frac{4}{3 + 5X}$$

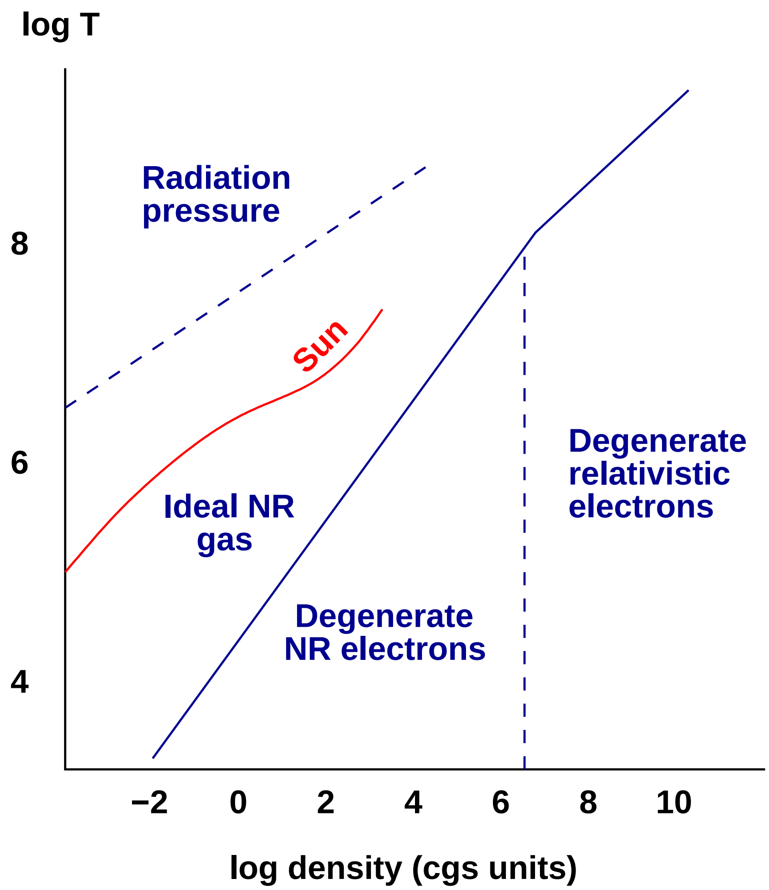
For a typical value of the hydrogen mass fraction $X = 0.7$, $\mu = 0.62$.

If in the center of a star all the H is converted to He, $X = 0$ and $\mu = 1.3$, i.e. conversion of all H $\rightarrow$ He changes μ by a factor of two.

The equation of state: Overview

In deriving the equation of state, need to consider,

- Quantum statistics (bosons vs fermions)
- Non-relativistic (NR) or relativistic particles
- Degenerate or non-degenerate



Onset of radiation pressure and degenerate relativistic matter both lead to $\gamma \rightarrow 4/3$ – set limits on the possible masses of stars / white dwarfs. Situation for neutron stars is more complicated.