Equation of state

In thermodynamic equilibrium, statistical mechanics gives the phase space density of a species of particle as:

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_j}{\exp[(\mathcal{E}_j + \mathcal{E}(p) - \mu)/kT] \pm 1}$$

where:

- $p = |\mathbf{p}|$ is the momentum. We will take n to be spherically symmetric in momentum space.
- g_j is the degeneracy of state j.
- $\mathcal{E}_j$ is the energy of state j relative to some reference level.
- $\mathcal{E}(p)$ is the kinetic energy.
- μ is the chemical potential:

  $$\mu \equiv \left(\frac{\partial E}{\partial N}\right)_{S,V}$$

  where E is the internal energy per unit mass, N is the number density per unit mass.
- The plus sign is for spin half particles (fermions), the negative sign for bosons with zero or integer spin.

From the phase space density, integrate to get quantities that appear in the equation of state:

(1) Physical space number density n (i.e. per cm^{-3}),

$$n = \int_p n(p) 4\pi p^2 dp$$

(2) Internal energy (note *unit volume* here),

$$E = \int_p n(p) \mathcal{E}(p) 4\pi p^2 dp$$

(3) Pressure,

$$P = \frac{1}{3} \int_p n(p) p v 4\pi p^2 dp$$

For relativistic particles the kinetic energy is,

$$\mathcal{E}(p) = (p^2 c^2 + m^2 c^4)^{1/2} - mc^2$$

with limits,

- $\mathcal{E}(p) = p^2/2m$ for nonrelativistic particles.
- $\mathcal{E}(p) = pc$ for highly relativistic particles.

Velocity v is,

$$v = \frac{\partial \mathcal{E}}{\partial p}.$$

Blackbody radiation

Photons have two polarization states ($g = 2$), zero chemical potential, and $\mathcal{E} = pc$. There are no excited states, so we may take $\mathcal{E}_j = 0$.

Integrals for pressure and internal energy give,

$$P_{\rm rad} = \left(\frac{8\pi^5}{15} \frac{k^4}{c^3 h^3} \right) \frac{T^4}{3} = \frac{1}{3} a T^4$$

$$E_{\rm rad} = a T^4$$

with the radiation constant $a = 7.566 \times 10^{-15}$ erg cm^{-3} K^{-4}.

Hence, $P = (\gamma - 1) \times$ energy per unit volume with $\gamma = 4/3$, as before.

Ideal monatomic gas

For an ideal gas, quantum nature of the constituent particles can be neglected. This requires (μ/kT) to be large and negative, so that the exponential dominates the denominator of the expression for the phase space density.

In this limit, consider a nonrelativistic gas,

- $\mathcal{E} = p^2/2m$
- $v = p/m$
- $\mathcal{E} = \mathcal{E}_0$ (single energy state)

The number density is then,

$$n = \frac{4\pi}{h^3} g \int_0^\infty p^2 e^{\mu/kT} e^{-\mathcal{E}_0/kT} e^{-p^2/2mkT} dp$$

which gives,

$$e^{\mu/kT} = \frac{nh^3}{g(2\pi mkT)^{3/2}} e^{\mathcal{E}_0/kT}$$

For a given density, use this expression to:

- Verify that $(\mu/kT) \ll 1$ (i.e. that we are justified in ignoring the quantum corrections). This obviously requires that $nT^{-3/2}$ be small.
- Determine μ.

Likewise, neglecting the ± 1 statistical factor gives the pressure as,

$$P = g \frac{4\pi}{h^3} \frac{\pi^{1/2}}{8m} (2mkT)^{5/2} e^{\mu/kT} e^{-\mathcal{E}_0/kT}$$

which upon substituting for $e^{\mu/kT}$ yields,

$$P = nkT.$$

The internal energy integral gives,

$$E = \frac{3}{2} nkT.$$

Degenerate material

Consider the equation of state of degenerate fermions (electrons are of the greatest interest here). If we take an energy reference level $\mathcal{E}_0 = mc^2$, then,

$$n = \frac{8\pi}{h^3} \int_0^\infty \frac{p^2 dp}{\exp[(mc^2 + \mathcal{E}(p) - \mu)/kT] + 1}$$

where we have used $g = 2$. In general,

$$\mathcal{E}(p) = mc^2 \left[\sqrt{1 + \left(\frac{p}{mc}\right)^2} - 1 \right]$$

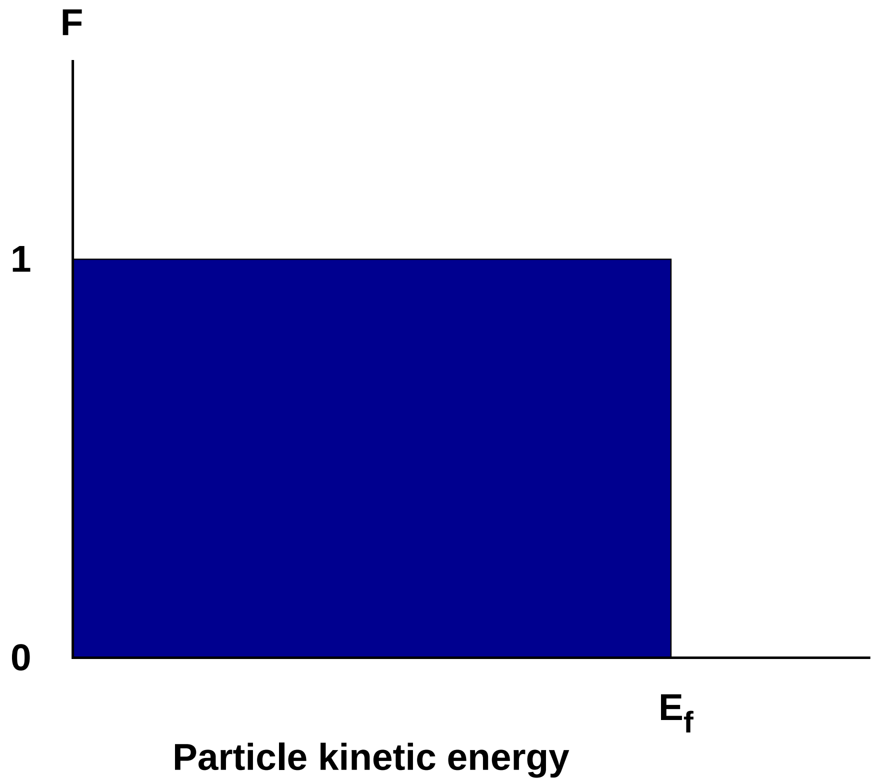
and,

$$v = \frac{p}{m} \left[1 + \left(\frac{p}{mc}\right)^2 \right]^{-1/2} .$$

For a completely degenerate gas, consider the limit as $T \to 0$. The probability of a state with energy $\mathcal{E}$ being occupied is,

$$F(\mathcal{E}) = \frac{1}{\exp[(\mathcal{E} - (\mu - mc^2))/kT] + 1}$$

with limits $F(\mathcal{E}) = 1$ for $\mathcal{E} < (\mu - mc^2)$ and $F(\mathcal{E}) = 0$ for $\mathcal{E} > (\mu - mc^2)$.



At zero T, all states are occupied up to the Fermi energy $\mathcal{E}_F$.

Corresponding Fermi momentum is p_F. Define a parameter x which measures how relativistic the gas is:

$$x \equiv \frac{p}{mc}$$

so that $x_F = p_F/mc$. Then,

$$\mathcal{E}_F = mc^2 \left[(1 + x_F^2)^{1/2} - 1 \right]$$

Note that at zero T the simple form of F means we don't need to explicitly consider the chemical potential.

Integrating for the number density:

$$n = \frac{8\pi}{h^3}\int_0^{p_F} p^2 dp = 8\pi \left(\frac{h}{mc}\right)^{-3} \int_0^{x_F} x^2 dx = \frac{8\pi}{3}\left(\frac{h}{mc}\right)^{-3} x_F^3$$

For electrons,

$$n_e = \frac{8\pi}{3}\left(\frac{h}{m_e c}\right)^{-3} x_F^3 = 5.9 \times 10^{29} x_F^3 \text{ cm}^{-3}$$

Writing $n_e = \rho N_A/\mu_e$, with μ_e the mean molecular weight in the gas per electron,

$$\frac{\rho}{\mu_e} \simeq 9.7 \times 10^5 \text{ g cm}^{-3} x_F^3$$

i.e. degenerate matter becomes relativistic at densities of order 10^6 g cm^{-3}.

Pressure is given by,

$$P_e = \frac{8\pi}{3}\frac{m_e^4 c^5}{h^3}\int_0^{x_F} \frac{x^4 dx}{(1+x^2)^{1/2}} = Af(x)$$

where the constant A is,

$$A = \frac{\pi}{3}\left(\frac{h}{m_e c}\right)^{-3} m_e c^2$$

The function $f(x)$ is,

$$f(x) = x(2x^2 - 3)(1 + x^2)^{1/2} + 3 \sinh^{-1} x$$

Likewise, the internal energy (units here erg cm^{-3}) is,

$$E_e = A g(x)$$

with,

$$g(x) = 8x^3[(1 + x^2)^{1/2} - 1] - f(x)$$

Limiting forms of these functions for $x \ll 1$ and $x \gg 1$ are given in Hansen & Kawaler §3.5.1.

Important point is:

$$P_e \propto E_e \propto \left(\frac{\rho}{\mu_e}\right)^{5/3}$$

for $x \ll 1$, while,

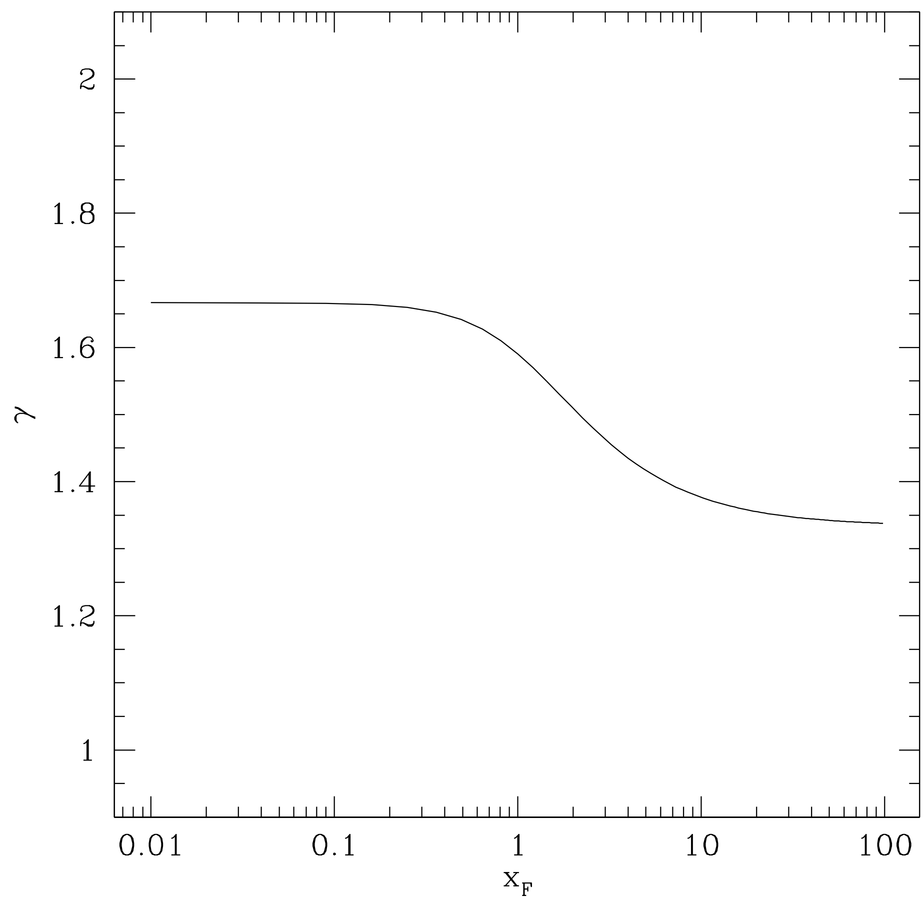
$$P_e \propto E_e \propto \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

for $x \gg 1$.

An effective γ can be defined as,

$$P_e = (\gamma - 1)E_e.$$

Using the expressions for f and g, the function $\gamma(x_F)$ has the form,



i.e. a completely degenerate electron gas has the same effective γ as an ideal monatomic gas, while in the relativistic limit it behaves like radiation pressure as far as γ is concerned.

Effects of finite temperature

As the temperature of the gas is raised, we expect that the degeneracy will be lifted once (to order of magnitude),

$$kT \approx \mathcal{E}_F$$

In the non-relativistic case, $\mathcal{E}_F = mc^2 x_F^2/2$, and converting to density one gets,

$$\frac{\rho}{\mu_e} \approx 6.0 \left(\frac{T}{10^6 \text{ K}} \right)^{3/2} \text{ g cm}^{-3}$$

If, at given T, the density exceeds this value, then the gas will be degenerate. Likewise in the relativistic regime,

$$\frac{\rho}{\mu_e} \approx 4.6 \times 10^6 \left(\frac{T}{10^{10} \text{ K}} \right)^{3} \text{ g cm}^{-3}$$

Boundary between the regimes falls below Solar values,