Thermodynamics of a gas + radiation pressure

Normally,

- Radiation pressure is important only for T high enough that the degree of ionization is high.
- Radiation pressure is usually unimportant in conditions where the electrons are degenerate.

Useful to consider case where pressure is provided by an ideal nondegenerate monatomic gas plus radiation pressure. Then,

$$P = P_g + P_r = \frac{N_A k}{\mu}\rho T + \frac{1}{3}aT^4$$

Internal energy (per gram) is,

$$U = \frac{N_A}{\mu}\left(\frac{3}{2}kT\right) + aT^4 V$$

where $V = 1/\rho$. For a quasistatic change the first law of thermodynamics is,

$$dQ = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV + PdV$$

(as before, except that now $U \neq U(T)$ only).

Evaluating the partial derivatives,

$$\left(\frac{\partial U}{\partial T}\right)_V = 4aT^3V + \frac{3}{2}\frac{N_Ak}{\mu}$$

$$\left(\frac{\partial U}{\partial V}\right)_T = aT^4$$

Then for this EOS the first law is,

$$dQ = \left(4aT^3V + \frac{3}{2}\frac{N_Ak}{\mu}\right)dT + \left(\frac{4}{3}aT^4 + \frac{N_Ak}{\mu}\frac{T}{V}\right)dV$$

For an adiabatic change, $dQ = 0$. We aim to express that equation in forms resembling those for an ideal gas,

$$\begin{aligned}
\frac{dP}{P} &+ \Gamma_1\frac{dV}{V} = 0 \\
\frac{dP}{P} &+ \frac{\Gamma_2}{1-\Gamma_2}\frac{dT}{T} = 0 \\
\frac{dT}{T} &+ (\Gamma_3 - 1)\frac{dV}{V} = 0
\end{aligned}$$

which defines the *adiabatic exponents*. These are not constants. Only two can be independent,

$$\Gamma_3 - 1 = (\Gamma_2 - 1)\frac{\Gamma_1}{\Gamma_2}.$$

Algebraic manipulations of the first law, the EOS, and these definitions yield the Γ's. e.g. for Γ_1:

From the equation of state,

$$\begin{aligned} dP &= \left(\frac{4}{3}aT^4 + \frac{N_A k}{\mu}\frac{T}{V}\right)\frac{dT}{T} - \frac{N_A k}{\mu}\frac{T}{V}\frac{dV}{V} \\ &= (4P_r + P_g)\frac{dT}{T} - P_g\frac{dV}{V} \end{aligned}$$

Substitute into definition of Γ_1,

$$(4P_r + P_g)\frac{dT}{T} + \left[\Gamma_1(P_r + P_g) - P_g\right]\frac{dV}{V} = 0$$

For $dQ = 0$, first law gives,

$$(12P_r + \frac{3}{2}P_g)\frac{dT}{T} + (4P_r + P_g)\frac{dV}{V} = 0.$$

Comparison of these expressions gives Γ_1 in terms of the partial pressures of gas and radiation,

$$\frac{\Gamma_1(P_r + P_g) - P_g}{4P_r + P_g} = \frac{4P_r + P_g}{12P_r + \frac{3}{2}P_g}$$

Matters are (slightly) simplified by defining the fraction of the total pressure contributed by the gas as β,

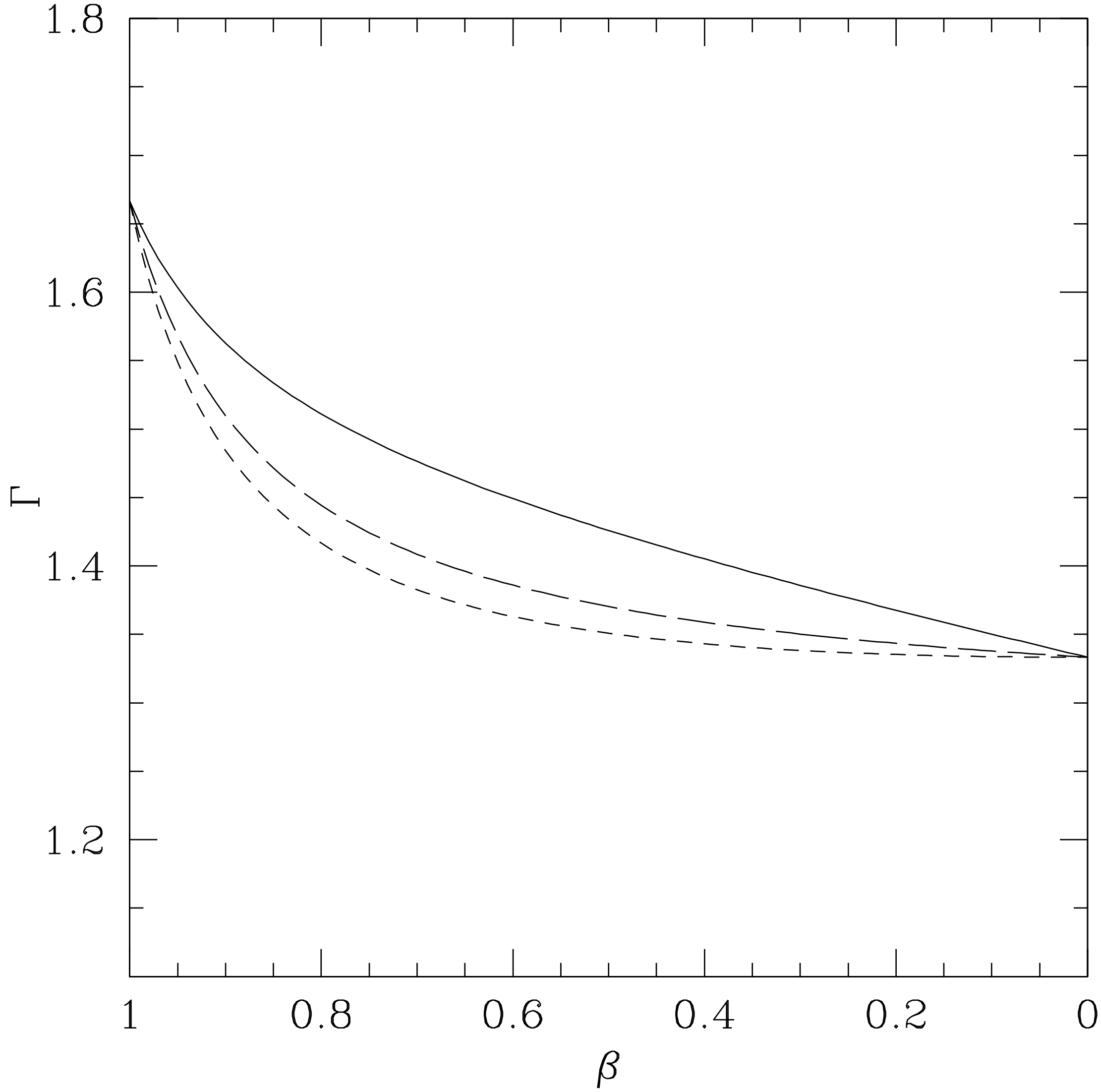
$$
\begin{aligned}
P_g &= \beta P \\
P_r &= (1-\beta) P
\end{aligned}
$$

(Note: not uncommon to see β defined instead as the ratio of the gas to radiation pressure...)

With this definition,

$$
\begin{aligned}
\Gamma_1 &= \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta} \\
\Gamma_2 &= \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2} \\
\Gamma_3 &= \frac{32 - 27\beta}{24 - 21\beta}
\end{aligned}
$$

All of these decrease monotonically from $5/3$ for $\beta = 1$ to $4/3$ for $\beta = 0$. Since they depend upon the state of the gas, the differential expressions are not immediately integrable as for an ideal gas.



- Γ_1: solid curve
- Γ_2: short dashed curve
- Γ_3: long dashed curve

Physical relevance of the different Γ's:

(1) The adiabatic sound speed v_s is given by,

$$v_s^2 = \left(\frac{dP}{d\rho}\right)_{ad} = \Gamma_1 \frac{P}{\rho}.$$

Hence Γ_1 is relevant dynamically – e.g. for pulsations.

(2) Γ_2 describes how the temperature responds to changes in pressure – determines whether the gas is unstable to convection. Useful to define the adiabatic gradient,

$$\frac{1}{\nabla_{ad}} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{ad} = \frac{\Gamma_2}{\Gamma_2 - 1}.$$

(3) Γ_3 describes most directly how the temperature responds to compression.

Specific heats

Specific heats of an ideal monatomic gas plus radiation are,

$$
\begin{aligned}
C_V = \left(\frac{dQ}{dT}\right)_V &= 4aT^3V + \frac{3}{2}\frac{N_A k}{\mu} \\
&= \frac{3}{2}\frac{N_A k}{\mu}\left(1 + \frac{8aT^4/3}{N_A k/\mu V}\right) \\
&= c_V\left(1 + \frac{8P_r}{P_g}\right) \\
&= c_V \frac{8 - 7\beta}{\beta}
\end{aligned}
$$

where $c_V = 3N_A k/2\mu$ is the specific heat at constant volume of the particle gas only.

Specific heat at constant pressure is,

$$
C_P = c_V \frac{\frac{32}{3} - 8\beta - \beta^2}{\beta^2}
$$

and the ratio is,

$$
\frac{C_P}{C_V} = \frac{\Gamma_1}{\beta}
$$

By writing the equations for differential changes in terms of the entropy, can derive an analgous expression to the $PV^\gamma = \text{const}$ which is valid for an ideal gas. Using $dQ = TdS$,

$$dS = \left(4aT^2V + \frac{3N_Ak}{2\mu T}\right)dT + \left(\frac{4}{3}aT^3 + \frac{N_Ak}{\mu V}\right)dV$$

Substitute $W = T^3V$,

$$dS = -\frac{3}{2}\frac{N_Ak}{\mu}\frac{dT}{T} + \frac{4a}{3}dW + \frac{N_Ak}{\mu}\frac{dW}{W}$$

which integrates to give,

$$S = \text{const} + \frac{N_Ak}{\mu}\ln\frac{T^{3/2}}{\rho} + \frac{4a}{3}\frac{T^3}{\rho}.$$

Note: additive terms for the entropy per unit mass of an ideal nondegenerate monatomic gas and of a photon gas.

Since $T^3/\rho \propto P_r/P_g$, can also write this as,

$$S = \text{const} + \frac{N_A k}{\mu}\left(\ln\frac{T^{3/2}}{\rho} + 4\frac{1-\beta}{\beta}\right)$$

The increase in entropy of the final state f over some initial state i is,

$$\Delta S = \frac{N_A k}{\mu}\left[\ln\left[\left(\frac{T_f}{T_i}\right)^{3/2}\frac{\rho_i}{\rho_f}\right] + 4\left(\frac{1-\beta_f}{\beta_f} - \frac{1-\beta_i}{\beta_i}\right)\right]$$

For a reversible change, $\Delta S = 0$, and the change is adiabatic.

Example: if a rising blob of gas in a convectively unstable zone of the star does not exchange heat with its surroundings, then this expression provides a relationship between the initial and final thermodynamic states.