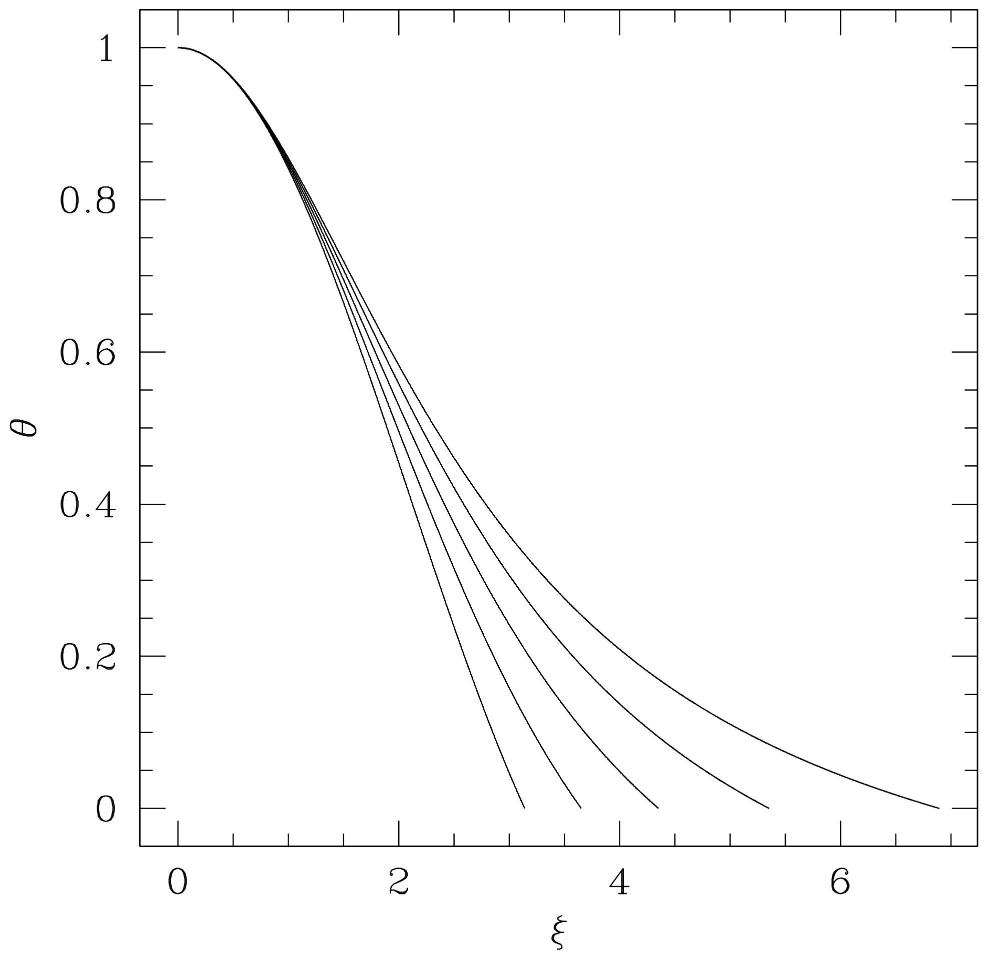
# Polytropic stellar models

Solutions to the Lane-Emden equation for $n = 1, 3/2, 2, 5/2, 3$:




Check of the numerical solutions: the $n = 1$ polytrope has $\xi_1 = \pi$, as it should.

As $n$ increases the solutions become less centrally concentrated.

## Application to stars

First, consider the case where $K$ is **not** fixed by the equation of state. We can construct a polytropic solution for specified $R$ and $M$, given an assumed $n$.

Mass within radius $r$ is:

$$m(r) = \int_0^r 4\pi r^2 \rho dr = 4\pi\rho_c \int_0^r \theta^n r^2 dr = 4\pi\rho_c \frac{r^3}{\xi^3} \int_0^\xi \theta^n \xi^2 d\xi$$

(taking $r^3/\xi^3$ outside the integral as it is a constant). From the Lane-Emden equation, the integrand is a derivative, so can immediately be integrated to give,

$$m(r) = 4\pi r^3 \rho_c \left( -\frac{1}{\xi}\frac{d\theta}{d\xi} \right)$$

At the surface $\xi = \xi_1$, the total mass is,

$$M = 4\pi R^3 \rho_c \left( -\frac{1}{\xi}\frac{d\theta}{d\xi} \right)_{\xi=\xi_1}$$

or, in terms of the mean density $\bar{\rho} = 3M/(4\pi R^3)$,

$$\frac{\bar{\rho}}{\rho_c} = \left( -\frac{3}{\xi}\frac{d\theta}{d\xi} \right)_{\xi=\xi_1}$$

The RHS is just a function of $n$, and can be tabulated or calculated numerically.

Numerical values for polytropes needed to compute the radius, mass, and central density concentration are:

| $n$ | $\xi_1$ | $\left(-\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1}$ | $\frac{\rho_c}{\bar{\rho}}$ |
|---|---|---|---|
| 0 | 2.45 | 4.90 | 1.00 |
| 1 | 3.14 | 3.14 | 3.29 |
| 1.5 | 3.65 | 2.71 | 5.99 |
| 2 | 4.35 | 2.41 | 11.4 |
| 3 | 6.90 | 2.02 | 54.2 |
| 4 | 15.0 | 1.80 | 622.4 |

To apply to a particular star it is just a matter of sorting out the scaling factors.

e.g. if the Sun is described by an $n = 3$ polytrope with mass $M_\odot = 1.989 \times 10^{33}$ g, radius $R_\odot = 6.96 \times 10^{10}$ cm,

$$\bar{\rho} = 1.41 \text{ g cm}^{-3}$$

implying for $n = 3$ a central density,

$$\rho_c = 76.3 \text{ g cm}^{-3}.$$

The scale factor $r_n = R_\odot / \xi_1 = 1.0 \times 10^{10}$. We therefore deduce that,

$$K = 3.8 \times 10^{14} \text{ cgs units}$$

so $P_c = 1.24 \times 10^{17}$ dynes cm$^{-2}$. For an ideal gas equation of state with $\mu = 0.62$,

$$T_c = 1.2 \times 10^7 \text{ K}.$$

A chemically homogenous stellar model with $M = 1\ M_\odot$ has $T_c = 1.4 \times 10^7$ K, so this approach gets us within $\approx 10\%$ of the correct answer.

Although we have focused on the central properties, note that the whole mechanical structure of the star is now specified once we have (numerically) calculated $\theta$ for $n = 3$.

## Radiation pressure

An $n = 3$ polytrope can also be appropriate for a star dominated by radiation pressure. As before, the equation of state is,

$$P = \frac{R}{\mu}\rho T + \frac{1}{3}aT^4 = \frac{R}{\mu\beta}\rho T.$$

Now, assume that $\beta = P_g/P$ is a constant throughout the star. Then,

$$1 - \beta = \frac{P_r}{P} = \frac{aT^4}{3P}$$

so, $\beta$ a constant implies that $T^4 \propto P$. Substituting into the equation of state,

$$P = \left(\frac{3R^4}{a\mu^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3} \rho^{4/3}$$

...a polytropic relation with $n = 3$ for constant $\beta$.

## Application to supermassive stars

For this equation of state, with $\beta$ constant,

$$K = \left(\frac{3R^4}{a\mu^4}\right)^{1/3} \left(\frac{1-\beta}{\beta^4}\right)^{1/3}.$$

For an $n = 3$ polytrope we also have,

$$K = \pi G \rho_c^{2/3} \frac{R^2}{\xi_1^2}.$$

The central density can be written as,

$$\rho_c = 54.2\bar{\rho} = 54.2\frac{3M}{4\pi R^3}$$

using the numerical values appropriate for $n = 3$. Equating the expressions for $K$ and eliminating $\rho_c$,

$$\frac{1-\beta}{\mu^4\beta^4} = 3.02 \times 10^{-3} \left(\frac{M}{M_\odot}\right)^2$$

which is 'Eddington's quartic'. Note:

- For $0 \leq \beta \leq 1$, the LHS is a monotonically decreasing function of $\beta$.
- This means that as $M$ increases, $\beta$ must decrease. Supermassive stars are dominated by radiation pressure.

## White dwarfs

For a degenerate gas at zero $T$, $K$ is fixed. This removes one degree of freedom compared to the previous case where $K$ could be taken as a free parameter.

Assume some central density $\rho_c$, and polytropic index $n$. Then $\rho = \rho_c \theta^n$ is a known function of $\xi$. The relation between $\xi$ and $r$ is

$$r = r_n \xi$$

where

$$r_n = \left[ \frac{(n+1)K}{4\pi G} \rho_c^{1/n-1} \right]^{1/2}.$$

At $r = R$, $\xi = \xi_1$, a known constant. The relation between central density and the model radius is,

$$R \propto \rho_c^{\frac{1-n}{2n}}.$$

For a given $n$,

$$M \propto \rho_c R^3.$$

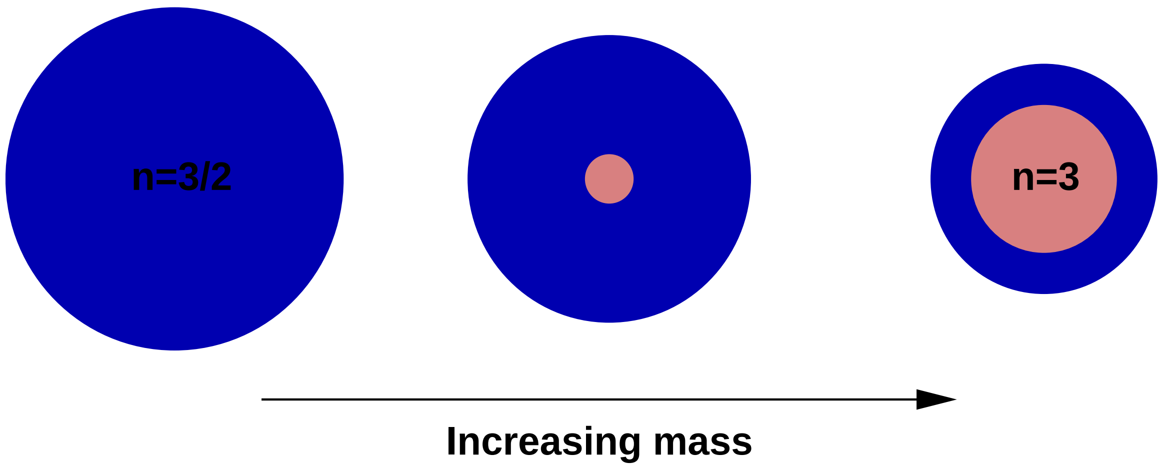
Eliminating $\rho_c$ gives a **mass-radius relation**:

$$R \propto M^{\frac{1-n}{3-n}}.$$

Consequences of this relation,

- There is now a one-dimensional family of models. We can freely specify *either* $M$ or $R$, but not both.
- For $n = 3/2$,
$$R \propto M^{-1/3},$$
i.e. the radius *shrinks* with increasing mass.
- With increasing mass, the central density rises. Eventually, this must invalidate the assumption that the electrons are non-relativistic. There will be a transition between $n = 3/2$ to $n = 3$.

Schematically, expect a relativistic core once $\rho_c \sim 10^6$ g cm$^{-3}$, growing larger with increasing mass. A complete model could be constructed by patching two polytropes, one with $n = 3$ and one with $n = 3/2$, smoothly together.



This breaks down once most of the star is relativistic. From the mass-central density relation, $M$ does not vary with $\rho_c$ for a polytrope with $n = 3$ if $K$ is fixed. Instead,

$$M = 4\pi \left( -\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1} \left( \frac{K}{\pi G} \right)^{3/2}.$$

Inserting numerical values for $K$, and the $n = 3$ value for the first bracket,

$$M_{Ch} = \frac{5.836}{\mu_e^2} M_\odot,$$

the **Chandrasekhar limiting mass** for a white dwarf. In practice, white dwarfs have compositions rich in helium, carbon and oxygen. Hence $\mu_e = 2$ and,

$$M_{Ch} \simeq 1.46\ M_\odot.$$

No white dwarf is observed with $M > M_{Ch}$. Many have masses well below the limit (e.g. $0.6\ M_\odot$). Interesting since even the merger of two low mass white dwarfs wouldn't exceed $M_{Ch}$.